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RADIANT HEAT TRANSFER IN AN ABSORBENT MEDIUM

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A dependence is obtained for the radiation flux vector in the form of a series. A calculation formula taking the anisotropy of the radiation field into account is proposed.

Various methods of calculating the radiant heat transfer in an absorbing (radiating) medium are based on a closed and in principle solvable system of differential equations containing the radiant-heat-transfer energy equation and the radiant-transfer equation [1]. For steady heat-transfer conditions, the energy equation may be written in the following form:

$$\operatorname{div} \mathbf{q}_r = 4\sigma(\varepsilon_* T^4 - \alpha T_r^4), \tag{1}$$

where T_r is the radiant temperature, defined at each point of the medium by the expression

$$T_r^4 = \frac{1}{4\sigma} \int\limits_{(4\pi)} I d\omega. \tag{2}$$

The radiation flux vector $\mathbf{q}_r$ may be found by vector integration over the spherical solid angle $\omega = 4\pi$ of the total radiation intensity I, determined from the radiation-transfer equation [1, 2] as follows:

$$I = \frac{\varepsilon_*}{\alpha} B - \frac{1}{\alpha} \frac{dI}{dl}. \tag{3}$$

The equilibrium radiation intensity B at each point of the volume of the medium is then calculated from the well-known formula [2]

$$B = \sigma T^4/\pi. \tag{4}$$

As a rule, the total radiation intensity I is not the same for different directions l, and its dependence on the solid angle ω is not known a priori. Therefore, the integration of Eq. (3) in general form is carried out only for the case of isotropic radiation [2].

It may be shown that on the basis of Eq. (3) calculational dependences may be obtained for the radiation flux vector with an arbitrary configuration of the absorbing-medium volume and an anisotropic radiation field. This involves differentiating repeatedly Eq. (3) with respect to the direction l, taking as constant the ratio between the intrinsic radiation and absorption coefficients of the medium ε_*/α, and neglecting, for simplicity of the equations, the derivatives of second and higher order of the absorption coefficient α. Note that on the right-hand side of each of the resulting equations there is a derivative of the total radiation intensity I of order one higher than that of the derivative of the total radiation intensity on the left-hand side of the equation. Using this structural property of the formulas, the derivative dI/dl may be eliminated from Eq. (3) and the total radiation intensity written as a uniformly converging power series,

$$I = \frac{\varepsilon_*}{\alpha} \left\{ B - \frac{1}{\varkappa_1 \alpha} \frac{dB}{dl} + \frac{1}{\varkappa_1 \varkappa_2 \alpha^2} \frac{d^2B}{dl^2} + \ldots + \frac{(-1)^n}{\varkappa_1 \varkappa_2 \ldots \varkappa_n \alpha^n} \frac{d^nB}{dl^n} + \ldots \right\}, \tag{5}$$

where $\varkappa_n = (1 + n d\alpha^{-1}/dl)$.

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At each point of the medium, a spherical coordinate system l, ψ, θ is established so that the direction l forms an angle φ with the z axis of the Cartesian coordinate system x, y, z and the elementary solid angle is defined by the equation

$$d\omega = \sin\varphi \cdot d\varphi \cdot d\theta.$$

Then the angles $(\widehat{x, l})$ and $(\widehat{y, l})$ between the x and y axes and the direction l may be found from the geometrical relations

$$\cos(\widehat{x, l}) = \sin\varphi \cdot \cos\theta, \quad \cos(\widehat{y, l}) = \sin\varphi \cdot \sin\theta.$$

Consider first of all the simplest case where the change in absorption coefficient may be neglected, which allows the derivative $d\alpha^{-1}/dl$ to be set equal to zero.

Replacing the derivatives with respect to l in Eq. (5) by derivatives with respect the coordinates x, y, and z, all the terms of the equation are then multiplied by $\cos\varphi$ and integrated over the spherical solid angle $\omega = 4\pi$, taking into account that the equilibrium radiation intensity B, the derivatives of B with respect to x, y, and z, and the coefficient ε_* do not depend on the solid angle ω and may be taken outside the integral sign. Taking the medium to be close to gray in its properties, the absorption coefficient α is also taken outside the integral sign. Replacing the equilibrium radiation intensity B by the temperature T^4 according to Eq. (4), the following expression is obtained for the projection of the radiation flux vector on the z axis:

$$q_{rz} = -4\sigma \frac{\varepsilon_*}{\alpha}\left\{\frac{1}{3\alpha}\frac{\partial T^4}{\partial z} + \frac{1}{5\alpha^3}\left(\frac{\partial^3 T^4}{\partial z^3} + \frac{\partial^3 T^4}{\partial x^2 \partial z} + \frac{\partial^3 T^4}{\partial y^2 \partial z}\right) + \right.$$
$$\left. + \frac{1}{7\alpha^5}\left(\frac{\partial^5 T^4}{\partial z^5} + \frac{\partial^5 T^4}{\partial x^4 \partial z} + \frac{\partial^5 T^4}{\partial y^4 \partial z} + 2\frac{\partial^5 T^4}{\partial x^2 \partial z^3} + 2\frac{\partial^5 T^4}{\partial y^2 \partial z^3} + 2\frac{\partial^5 T^4}{\partial x^2 \partial y^2 \partial z}\right) + \dots\right\}. \quad (6)$$

Analogous expressions may be obtained for the projections of the radiation flux vector q_{rx}, q_{ry} on the x and y axes by cyclical replacement of variables ($x \to y$, $y \to z$, $z \to x$, $q_{rz} \to q_{rx}$, $q_{rx} \to q_{ry}$) in Eq. (6).

The resulting equations may be regarded as the series expansion of the radiation vector projection in terms of α^{-1}, the reciprocal of the absorption coefficient. In the particular case when the gradients of the temperatures T and T_r are in the same direction, and the intrinsic-radiation coefficient ε_* is equal to the absorption coefficient α, the first term of the expansion in Eq. (6) gives an expression coinciding with the well-known Rosseland approximation [2, 4]:

$$q_r = -\frac{4\sigma}{3\alpha}\,\mathrm{grad}\,T^4. \quad (7)$$

Thus, Eq. (6) differs from the Rosseland approximation in Eq. (7) by terms containing higher-order derivatives which take into account the effect of radiation-field anisotropy on the resulting radiation fluxes. It is of interest to estimate the magnitudes of these terms for real absorbing media.

According to measurements of the anisotropic radiation field in a small-quantity heating chamber of layer type [3], the maximum deviation of the actual thermal flux from that calculated from the formula for isotropic radiation is 15%, with a mean deviation of 1.5%. Since increase in order of the derivatives is accompanied by a rise (as $\alpha^{-1} \to 0$) in the order of smallness of the terms in the expansion containing these derivatives, it may be assumed on the basis of the given experimental data that in real conditions the terms in Eq. (6) with fifth-order derivatives amount to no more than 3-5% of the density of the resulting radiation flux. Then the terms containing derivatives of seventh and higher order may be neglected.

Integrating Eq. (5) over the spherical solid angle $\omega = 4\pi$, and bearing in mind Eq. (2), an equation relating the radiant and thermodynamic temperatures is obtained:

$$T^4 = \frac{\varepsilon_*}{\alpha}\left\{T^4 + \frac{1}{3\alpha^2}\left(\frac{\partial^2 T^4}{\partial x^2} + \frac{\partial^2 T^4}{\partial y^2} + \frac{\partial^2 T^4}{\partial z^2}\right) + \frac{1}{5\alpha^4}\left(\frac{\partial^4 T^4}{\partial x^4} + \frac{\partial^4 T^4}{\partial y^4} + \frac{\partial^4 T^4}{\partial z^4} + \right.\right.$$
$$\left.\left. + 2\frac{\partial^4 T^4}{\partial x^2 \partial y^2} + 2\frac{\partial^4 T^4}{\partial y^2 \partial z^2} + 2\frac{\partial^4 T^4}{\partial z^2 \partial x^2}\right) + \dots\right\}. \quad (8)$$

Taking from the terms of this equation the derivatives with respect to z, the term containing the first-order derivative is expressed in explicit form and substituted into Eq. (6) to give a new expression for the projection of the radiation flux vector on the z axis:

$$q_{rz} = -\frac{4\sigma}{3\alpha}\frac{\partial T_r^4}{\partial z} - 4\sigma\frac{\varepsilon_*}{\alpha}\left\{\frac{4}{45\alpha^3}\left(\frac{\partial^3 T^4}{\partial z^3} + \frac{\partial^3 T^4}{\partial x^2 \partial z} + \frac{\partial^3 T^4}{\partial y^2 \partial z}\right) + \right.$$
$$\left. + \frac{8}{105\alpha^5}\left(\frac{\partial^5 T^4}{\partial z^5} + \frac{\partial^5 T^4}{\partial x^4 \partial z} + \frac{\partial^5 T^4}{\partial y^4 \partial z} + 2\frac{\partial^5 T^4}{\partial x^2 \partial z^3} + 2\frac{\partial^5 T^4}{\partial y^2 \partial z^3} + 2\frac{\partial^5 T^4}{\partial x^2 \partial y^2 \partial z}\right) + \ldots\right\}. \tag{9}$$

Expressions for q_{rx} and q_{ry} may be obtained from Eq. (9) by cyclical substitution of variables. Retaining only the first term on the right-hand sides of the resulting equations, the well-known gradient representation of the radiation flux vector is obtained [3]:

$$\mathbf{q}_r = -\frac{4\sigma}{3\alpha}\operatorname{grad} T_r^4. \tag{10}$$

Comparing the numerical coefficients in Eqs. (6) and (9) for the terms containing higher-order derivatives, it may be concluded that Eq. (10) gives a better approximation than Eq. (7).

It may be shown further that it is possible to replace terms of Eq. (9) with higher-order derivatives by differences of first-order derivatives. Subtracting Eq. (9) from Eq. (6) gives the result

$$\frac{1}{3\alpha}\left(\frac{\partial T_r^4}{\partial z} - \frac{\varepsilon_*}{\alpha}\frac{\partial T^4}{\partial z}\right) = \frac{\varepsilon_*}{\alpha}\left\{\frac{1}{9\alpha^3}\left(\frac{\partial^3 T^4}{\partial z^3} + \frac{\partial^3 T^4}{\partial x^2 \partial z} + \right.\right.$$
$$\left.\left. + \frac{\partial^3 T^4}{\partial y^2 \partial z}\right) + \frac{1}{15\alpha^5}\left(\frac{\partial^5 T^4}{\partial z^5} + \frac{\partial^5 T^4}{\partial x^4 \partial z} + \frac{\partial^5 T^4}{\partial y^4 \partial z} + 2\frac{\partial^5 T^4}{\partial x^2 \partial z^3} + 2\frac{\partial^5 T^4}{\partial y^2 \partial z^3} + 2\frac{\partial^5 T^4}{\partial x^2 \partial y^2 \partial z}\right) + \ldots\right\}. \tag{11}$$

It is evident from a comparison of the right-hand sides of Eqs. (9) and (11) that an approximate calculational estimate of the effect of radiation-field anisotropy on the resulting heat flux at any point of the absorbing-medium volume may be obtained using the difference in the gradients of the fourth powers of the radiant and thermodynamic temperatures.

Next, Eq. (11) is multiplied by 4/5 and substituted into Eq. (9). Taking into account the analogous relations for the x and y axes, the dependence for the radiation flux vector is written in the following calculational form:

$$\mathbf{q}_r \approx -\frac{12\sigma}{5\alpha}\operatorname{grad}\left(T_r^4 - \frac{4\varepsilon_*}{9\alpha}T^4\right). \tag{12}$$

As follows from a comparison of the numerical coefficients in Eqs. (11) and (9), the difference in the first derivatives (gradients) of the fourth powers of the radiant and thermodynamic temperatures completely replaces, as a result of substitution in Eq. (9), the terms of this equation with third-order derivatives, and replaces by as much as 70% the terms with fifth-order derivatives, which gives Eq. (12) an accuracy sufficient for engineering calculations.

Now consider, for the example of a one-dimensional problem, how the temperature dependence of the absorption coefficient [2, 4] affects the form of the calculational equations. Direct integration of Eq. (5) leads to complex formulas considerably different from those obtained above. Therefore, assuming that the gradients of α^{-1} and the thermodynamic temperature coincide in direction with the z axis of the Cartesian coordinate system, the derivative with respect to the direction, $d\alpha^{-1}/dl$, is replaced by the derivative with respect to z and, introducing the notation $k = \partial\alpha^{-1}/\partial z$ for the sake of simplicity, the coefficients $1/\varkappa_1\varkappa_2 \ldots \varkappa_n$ in Eq. (5) are expanded in Taylor series with respect to k:

$$I = \frac{\varepsilon_*}{\alpha}B - (1 - k\cos\varphi + k^2\cos^2\varphi - k^3\cos^3\varphi + \ldots)\frac{\varepsilon_*}{\alpha^2}\frac{dB}{dl} +$$
$$+ (1 - 3k\cos\varphi + 7k^2\cos^2\varphi - 15k^3\cos^3\varphi + 31k^4\cos^4\varphi -$$
$$- 63k^5\cos^5\varphi + \ldots)\frac{\varepsilon_*}{\alpha^3}\frac{d^2B}{dl^2} - (1 - 6k\cos\varphi + 25k^2\cos^2\varphi -$$
$$- 90k^3\cos^3\varphi + 301k^4\cos^4\varphi - \ldots)\frac{\varepsilon_*}{\alpha^4}\frac{d^3B}{dl^3} + \ldots \tag{13}$$

Integrating over the spherical solid angle, and making transformations analogous to those above, dependences are obtained for the resulting radiation flux density:

$$q_r = -\frac{4\sigma}{3\alpha}\frac{\partial T_r^4}{\partial z} - 4\sigma\frac{\varepsilon_*}{\alpha}\left\{\left(\frac{4}{45}k^2 + \frac{8}{105}k^4 + \ldots\right)\frac{1}{\alpha}\frac{\partial T^4}{\partial z} + \right.$$
$$\left. + \left(\frac{4}{15}k + \frac{8}{7}k^3 + 4k^5 + \ldots\right)\frac{1}{\alpha^2}\frac{\partial^2 T^4}{\partial z^2} + \left(\frac{4}{45} + \frac{40}{21}k^2 + \frac{172}{9}k^4 + \ldots\right)\frac{1}{\alpha^3}\frac{\partial^3 T^4}{\partial z^3} + \ldots\right\} \quad (14)$$

and for the difference in gradients of the fourth powers of the radiant and thermodynamic temperatures:

$$\frac{1}{3\alpha}\left(\frac{\partial T_r^4}{\partial z} - \frac{\varepsilon_*}{\alpha}\frac{\partial T^4}{\partial z}\right) = \frac{\varepsilon_*}{\alpha}\left\{\left(\frac{1}{9}k^2 + \frac{1}{15}k^4 + \ldots\right)\frac{1}{\alpha}\frac{\partial T^4}{\partial z} + \right.$$
$$\left. + \left(\frac{1}{3}k + k^3 + 3k^5 + \ldots\right)\frac{1}{\alpha^2}\frac{\partial^2 T^4}{\partial z^2} + \left(\frac{1}{9} + \frac{5}{3}k^2 + \frac{43}{3}k^4 + \ldots\right)\frac{1}{\alpha^3}\frac{\partial^3 T^4}{\partial z^3} + \ldots\right\}. \quad (15)$$

It is not difficult to establish that in this case too it is possible using Eq. (15) to replace sufficiently completely the terms with higher-order derivatives in Eq. (14) by the difference in the gradients of the fourth powers of the radiant and thermodynamic temperatures and to obtain a calculational dependence coinciding (at least for $k \le 0.1$) with Eq. (12).

The possibility of using the resulting formulas for large values of the derivative $\partial\alpha^{-1}/\partial z$ evidently requires further verification.

NOTATION

B, I	are the equilibrium and total radiation intensities;
T, T_r	are the thermodynamic and radiant temperatures;
$\mathbf{q}_r$, q_r	are the radiation flux vector and flux density of the resulting radiation (modulus of radiation flux vector);
q_{rx}, q_{ry}, q_{rz}	are the projections of radiation vector;
n	is the number of natural series;
x, y, z, l	are the coordinate axes;
σ	is the Stefan–Boltzmann constant;
α, ε_*	are the absorption and intrinsic radiation coefficients of medium;
φ, θ	are the angles of spherical coordinate system;
ω	is the solid angle.

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